

EXPONENTIAL SUMS AND POLYNOMIAL CONGRUENCES ALONG p -ADIC SUBMANIFOLDS

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ABSTRACT. In this article, we consider the estimation of exponential sums along the points of the reduction mod p^m of a p -adic analytic submanifold of \mathbb{Z}_p^n . More precisely, we extend Igusa's stationary phase method to this type of exponential sums. We also study the number of solutions of a polynomial congruence along the points of the reduction mod p^m of a p -adic analytic submanifold of \mathbb{Z}_p^n . In addition, we attach a Poincaré series to these numbers, and establish its rationality. In this way, we obtain geometric bounds for the number of solutions of the corresponding polynomial congruences.

1. INTRODUCTION

Let K be a p -adic field, i.e. $[K : \mathbb{Q}_p] < \infty$. Let R_K be the valuation ring of K , P_K the maximal ideal of R_K , and $\overline{K} = R_K/P_K$ the residue field of K . The cardinality of the residue field of K is denoted by q , thus $\overline{K} = \mathbb{F}_q$. For $z \in K$, $\text{ord}(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation, $|z|_K = q^{-\text{ord}(z)}$ the p -adic norm, and $acz = z\pi^{-\text{ord}(z)}$ the angular component of z , where π is a fixed uniformizing parameter of R_K .

Let $f_i \in K[[x_1, \dots, x_n]]$ be a formal power series for $i = 1, \dots, l$, with $l \geq 2$, and put $x = (x_1, \dots, x_n)$. Let U be an open and compact subset of K^n . Assume that each series f_i converges on U . We set

$$V^{(l-1)} := V^{(l-1)}(K) := \{x \in U \mid f_1(x) = \dots = f_{l-1}(x) = 0\}$$

and assume that $V^{(l-1)}$ is a non-empty closed submanifold of U , with dimension $m := n - l + 1 \geq 1$, which implies that $n \geq l$. We assume that f_l is not identically zero on $V^{(l-1)}$ and that f_l has a zero on $V^{(l-1)}$. We consider on $V^{(l-1)}$ an analytic differential form Θ of degree m , and denote the measure induced on $V^{(l-1)}$ as $|\Theta|$. Later on, we specialize Θ to a Gel'fand-Leray form γ_{GL} on $V^{(l-1)}$. Let $\Phi : K^n \rightarrow \mathbb{C}$ be a Bruhat-Schwartz function with support in U . Let $\omega \in \Omega_0(K^\times)$ be a quasicharacter of K^\times , see Section 2.1. To these data we associate the following local zeta function:

$$\begin{aligned} Z_\Phi(\omega, V^{(l-1)}, f_l) &:= Z_\Phi(\omega, f_1, \dots, f_l, \Theta) \\ &:= \int_{V^{(l-1)}(K) \setminus f_l^{-1}(0)} \Phi(x) \omega(f_l(x)) |\Theta|. \end{aligned}$$

This function is holomorphic on $\Omega_0(K^\times)$, and has a meromorphic continuation to the whole $\Omega(K^\times)$ as a rational function of $t = \omega(\pi) = q^{-s}$. The real parts of the

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poles of the meromorphic continuation are negative rational numbers. These assertions follow directly from Igusa's results from the case $V^{(l-1)}(K) = K^{n-l+1} \cap U$, [16, Chap. 8], alternatively, see [29, Proposition 2.2]. In section 3, by using a suitable version of Hironaka's resolution theorem (see Theorem 1), we give a list of candidate poles of the local zeta function in terms of a certain embedded resolution (see Theorem 2). The purpose of this paper is sharpening some results of [29], mainly those connecting the poles of local zeta functions of type $Z_\Phi(\omega, V^{(l-1)}, f_l, \Theta)$ with the estimation of exponential sums and with the number of solutions of polynomial congruences along p -adic submanifolds, see Theorems 3, 4, 5. We also answer a question posed in [29] about $Z_\Phi(\omega, V^{(l-1)}, f_l, \gamma_{GL})$ and the Dirac delta function (see Theorem 6).

The zeta functions $Z_\Phi(\omega, V^{(l-1)}, f_l, \gamma_{GL})$ were studied in [29], non-Archimedean case, and in [13], Archimedean case with $l = 2$, when f_1, \dots, f_l are non-degenerate with respect to their Newton polyhedra, see also [11]. But if $V^{(l-1)}$ is replaced by K^n , the corresponding zeta functions have been extensively studied by Weil, Tate, Igusa, Denef, Loeser, among others, see e.g. [6], [7], [15], [16].

In [29, Theorem 4.8], Igusa's method for estimating exponential sums was extended to exponential sums of type:

$$E(z) := q^{-m(n-l+1)} \sum_{\overline{x} \in V^{(l-1)}(R_K) \bmod P_K^m} \Psi(z f_l(x)),$$

where $|z|_K = q^m$ with $m \in \mathbb{N}$, $\Psi(\cdot)$ is an additive character of K , and $V^{(l-1)}$ is a p -adic submanifold of R_K^n with 'good reduction mod P_K ' and ' f_1, \dots, f_l are non-degenerate with respect to their Newton polyhedra.' Our main result, and also the main motivation for this paper, is an extension of Igusa's stationary phase method to exponential sums of type $E(z)$ without the two above-mentioned conditions, see Theorem 4. At this point, it is worth to mention that the exponential sums along varieties over finite fields have been extensively studied [1], [4], [5], [9], [12], [22], [23], among others. For exponential sums mod p^m , we can mention the references [3], [6], [15], [16], [18], [19], [20], [27], [29], among others. The problem of extending Igusa's stationary phase method to exponential sums along 'varieties mod p^m ' was posed by Moreno in [20].

A more general problem is to estimate oscillatory integrals of type

$$E_\Phi(z, V^{(l-1)}, f_l, \Theta) := \int_{V^{(l-1)}(K)} \Phi(x) \Psi(z f_l(x)) |\Theta|,$$

for $|z|_K \gg 0$. The relevance of studying integrals of type $E_\Phi(z, V^{(l-1)}, f_l, \Theta)$ was pointed out in [17] by Kazhdan. In this paper we extend Igusa's method to oscillatory integrals of type $E_\Phi(z, V^{(l-1)}, f_l, \Theta)$, more precisely, we show the existence of an asymptotic expansion for $E_\Phi(z, V^{(l-1)}, f_l, \Theta)$, $|z|_K \gg 0$, which is controlled by the poles of $Z_\Phi(\omega, V^{(l-1)}, f_l, \Theta)$ (see Theorem 3).

We also consider the Poincaré series associated to the number of solutions of polynomial congruences along a p -adic submanifold of R_K^n (see Section 6). We show the rationality of a such Poincaré series and obtain a bound for the number of solutions of these polynomial congruences, see Theorem 5, Remark 3, and [6], [8], [16].

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2. PRELIMINARIES

2.1. Quasicharacters and local zeta functions. A quasicharacter of K^\times is a continuous group homomorphism from K^\times into \mathbb{C}^\times . The set of quasicharacters forms an Abelian group denoted as $\Omega(K^\times)$. We set $\omega_s(z) := |z|_K^s$ for $s \in \mathbb{C}$ and $z \in K^\times$, thus $\omega_s \in \Omega(K^\times)$. Let now $\omega \in \Omega(K^\times)$. If we choose $s \in \mathbb{C}$ satisfying $\omega(\pi) = q^{-s}$, then $\omega(z) = \omega_s(z) \chi(ac z)$ in which $\chi := \omega|_{R_K^\times}$ is a character of R_K^\times , i.e. a continuous group homomorphism from R_K^\times into the unit circle of the complex plane. Hence $\Omega(K^\times)$ is a one dimensional complex manifold since $\Omega(K^\times) \cong \mathbb{C} \times (R_K^\times)^*$, where $(R_K^\times)^*$ is the group of characters of R_K^\times . We note that $\sigma(\omega) := \operatorname{Re}(s)$ depends only on ω , and $|\omega(z)| = \omega_{\sigma(\omega)}(z)$. We define for every $\sigma \in \mathbb{R}$ an open subset of $\Omega(K^\times)$ by

$$\Omega_\sigma(K^\times) := \{\omega \in \Omega(K^\times) \mid \sigma(\omega) > \sigma\}.$$

For further details we refer the reader to [16].

Let $f_i \in K[[x_1, \dots, x_n]]$, $i = 1, \dots, l$, with $l \geq 2$, U , $V^{(l-1)}$, and Φ be as in the introduction. We consider on $V^{(l-1)}$, a closed submanifold of dimension m , an analytic differential form Θ of degree m , and denote the measure induced on $V^{(l-1)}$ as $|\Theta|$. We refer the reader to [24] and [21] for further details on p -adic manifolds and analytic subsets. Later on, we specialize Θ to a Gel'fand-Leray form γ_{GL} on $V^{(l-1)}$, i.e. a form satisfying $\gamma_{GL} \wedge \wedge_{i=1}^{l-1} df_i = \wedge_{i=1}^n dx_i$. The Gel'fand-Leray form is not unique, but its restriction to $V^{(l-1)}$ is independent of the choice of γ_{GL} (see [11, Chap. III, Sect. 1-9]). By passing to a sufficiently fine covering of the support of Φ , $Z_\Phi(\omega, V^{(l-1)}, f_l)$ can be expressed as a finite sum of classical Igusa's zeta functions, in this way one verifies that $Z_\Phi(\omega, V^{(l-1)}, f_l)$ is holomorphic on $\omega \in \Omega_0(K^\times)$. Since any $\omega \in \Omega(K^\times)$ can be expressed as $\omega(z) = \chi(ac z) |z|_K^s$, we use the notation $Z_\Phi(s, \chi) := Z_\Phi(s, \chi, V^{(l-1)}, f_l) := Z_\Phi(\omega, V^{(l-1)}, f_l)$.

2.2. Resolution of singularities. The following version of Hironaka's resolution theorem will be used later on:

Theorem 1 (Hironaka, [14]). *There exists an embedded resolution $\sigma : Y \rightarrow V^{(l-1)}$ of $f_l : V^{(l-1)} \rightarrow K$ and Θ , that is,*

- (1) *Y is an m -dimensional K -analytic compact manifold, and σ is a proper K -analytic map which is an isomorphism outside of $S := f_l^{-1}(0)$;*
- (2) *$\sigma^{-1}(S) = \cup_{i \in T} E_i$, where the E_i are closed submanifolds of Y of codimension one, each equipped with a pair of positive integers (N_i, v_i) satisfying the following: at every point b of Y there exist local coordinates (y_1, \dots, y_m) on Y around b such that, if E_1, \dots, E_k are the E_i containing b , we have on some neighborhood of b that E_i is given by $y_i = 0$ for $i = 1, \dots, k$,*

$$f_l \circ \sigma = \varepsilon(y) \prod_{i=1}^k y_i^{N_i},$$

and

$$\sigma^* \Theta = \eta(y) \left(\prod_{i=1}^k y_i^{v_i-1} \right) dy_1 \wedge \dots \wedge dy_m,$$

where $\varepsilon(y), \eta(y)$ are units in the local ring of Y at b .

The above theorem is a variation of Theorem 2.2 in [10], and its proof follows from Corollary 3 in [14] by using the reasoning given at the bottom of p. 97 in [10].

We call the $(N_i, v_i), i \in T$, the numerical data of (σ, Θ) . From now on, we fix Θ and say that $(N_i, v_i), i \in T$, are the numerical data of $\sigma : Y \rightarrow V^{(l-1)}$. Set $\rho := \rho(V^{(l-1)}, f_l, \Theta) := \min_{i \in T} v_i/N_i$.

We denote the set of critical points of the map $f_l : V^{(l-1)} \rightarrow K$ by C_{f_l} , i.e.

$$\begin{aligned} C_{f_l} &= \left\{ x \in V^{(l-1)} \mid \text{rank}_K \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} \leq l-1 \right\} \\ &= \left\{ x \in V^{(l-1)} \mid \text{rank}_K \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} = l-1 \right\}, \end{aligned}$$

since $\text{rank}_K \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{1 \leq i \leq l-1, 1 \leq j \leq n} = l-1$ for $x \in V^{(l-1)}$. For $\lambda \in K^\times$, we set $V^{(l, \lambda)} := \{x \in U \mid f_1(x) = \dots = f_{l-1}(x) = 0, f_l(x) = \lambda\}$.

Remark 1. Note that the following three statements are equivalent: (1) $C_{f_l} \subset f_l^{-1}(0)$; (2) for every $\lambda \in K^\times$, the l -form $\bigwedge_{i=1}^l df_i$ does not vanish on $V^{(l, \lambda)}$, and (3) for every $\lambda \in K^\times$, $V^{(l, \lambda)}$ is a closed submanifold of dimension $n-l$.

3. POLES OF LOCAL ZETA FUNCTIONS SUPPORTED ON p -ADIC SUBMANIFOLDS

Theorem 2. Let $\sigma : Y \rightarrow V^{(l-1)}$ be a fixed embedded resolution of $f_l : V^{(l-1)} \rightarrow K$, with numerical data $(N_i, v_i), i \in T$. Then

(1) $Z_\Phi(s, \chi, V^{(l-1)}, f_l)$ has a meromorphic continuation as a rational function of q^{-s} . Its poles are among the values

$$s = -\frac{v_i}{N_i} - \frac{2\pi\sqrt{-1}}{N_i \log q} k, \text{ with } k \in \mathbb{Z}, \text{ and } i \in T,$$

such that the order of χ divides N_i ;

(2) if $C_{f_l} \subset f_l^{-1}(0)$, then there exists $e(\Phi) > 0$ in \mathbb{N} such that

$$Z_\Phi(s, \chi, V^{(l-1)}, f_l) = 0 \quad \text{for every } s \in \mathbb{C},$$

unless the conductor $c(\chi)$ of χ satisfies $c(\chi) \leq e(\Phi)$;

(3) $-\rho$ is the real part of a pole of $Z_\Phi(s, \chi_{\text{triv}}, V^{(l-1)}, f_l)$, for some Φ with support in U , and consequently, ρ is independent of the embedded resolution chosen.

Proof. (1) The proof uses the same argument of the case $V^{(l-1)}(K) = K^{n-l+1} \cap U$, see [16, Theorem 8.2.1]. (2) The proof is a variation of the one given for Theorem 4.4 in [29].

(3) The proof is analogous to the one given for Theorem 2.7 in [25]. \square

Remark 2. We set $m_\rho := m_\rho(V^{(l-1)}, f_l, \Theta)$ for the largest multiplicity of the poles of $(1 - q^{-1-s})Z_\Phi(s, \chi_{\text{triv}}, V^{(l-1)}, f_l, \Theta)$ and $Z_\Phi(s, \chi, V^{(l-1)}, f_l, \Theta)$ with $\chi \neq \chi_{\text{triv}}$ having real part $-\rho$, when Φ runs through all the Bruhat-Schwartz functions. Note that by the previous theorem ρ and m_ρ are well-defined.

4. THE OSCILLATORY INTEGRALS $E_\Phi(z)$

4.1. Additive characters. Given $z = \sum_{n=n_0}^{\infty} z_n p^n \in \mathbb{Q}_p$, with $z_n \in \{0, \dots, p-1\}$ and $z_{n_0} \neq 0$, we set

$$\{z\}_p := \begin{cases} 0 & \text{if } n_0 \geq 0 \\ \sum_{n=n_0}^{-1} z_n p^n & \text{if } n_0 < 0, \end{cases}$$

the fractional part of z . Then $\exp(2\pi\sqrt{-1}\{z\}_p)$, $z \in \mathbb{Q}_p$, is an additive character on \mathbb{Q}_p trivial on \mathbb{Z}_p but not on $p^{-1}\mathbb{Z}_p$.

We recall that there exists an integer $d \geq 0$ such that $Tr_{K/\mathbb{Q}_p}(z) \in \mathbb{Z}_p$ for $|z|_K \leq q^d$ but $Tr_{K/\mathbb{Q}_p}(z_0) \notin \mathbb{Z}_p$ for some z_0 with $|z_0|_K = q^{d+1}$. The integer d is called the *exponent of the different* of K/\mathbb{Q}_p . It is known that $d \geq e-1$, where e is the ramification index of K/\mathbb{Q}_p , see e.g. [26, Chap. VIII, Corollary of Proposition 1]. The additive character

$$\varkappa(z) = \exp(2\pi\sqrt{-1}\{Tr_{K/\mathbb{Q}_p}(\pi^{-d}z)\}_p), \quad z \in K,$$

is a *standard character* of K , i.e. \varkappa is trivial on R_K but not on P_K^{-1} . For our purposes, it is more convenient to use

$$\Psi(z) = \exp(2\pi\sqrt{-1}\{Tr_{K/\mathbb{Q}_p}(z)\}_p), \quad z \in K,$$

instead of $\varkappa(\cdot)$. This particular choice is due to the fact that we use Denef's approach for estimating oscillatory integrals, see [6, Proposition 1.4.4].

4.2. Asymptotic expansion of oscillatory integrals. We set

$$E_\Phi(z) := E_\Phi(z, V^{(l-1)}, f_l, \Theta) = \int_{V^{(l-1)}(K)} \Phi(x) \Psi(z f_l(x)) |\Theta|,$$

for $z \in K$. The following theorem describes the asymptotic behavior of oscillatory integrals $E_\Phi(z)$.

Theorem 3. *Let $\sigma : Y \rightarrow V^{(l-1)}$ be any embedded resolution of $f_l : V^{(l-1)} \rightarrow K$, with numerical data (N_i, v_i) , $i \in T$. Assume that $C_{f_l} \subset f_l^{-1}(0)$. Then*

(1) *for $|z|_K$ big enough $E_\Phi(z)$ is a finite \mathbb{C} -linear combination of functions of the form $\chi(ac z) |z|_K^\lambda (\log_q |z|_K)^\beta$ with coefficients independent of z , and with $\lambda \in \mathbb{C}$ a pole of $(1 - q^{-1-s}) Z_\Phi(s, \chi_{\text{triv}}, V^{(l-1)}, f_l)$ or of $Z_\Phi(s, \chi, V^{(l-1)}, f_l)$, and $\beta \in \mathbb{N}$, with $\beta \leq (\text{multiplicity of } \lambda) - 1$. In addition, all poles λ appear effectively in this linear combination.*

(2) *There exists a constant C such that for $|z|_K > 1$,*

$$|E_\Phi(z)| \leq C |z|_K^{-\rho} (\log_q |z|_K)^{m_\rho - 1}.$$

Proof. (1) Let $\text{Coeff}_{t^k} Z_\Phi(s, \chi)$ denote the coefficient of t^k in the power expansion of $Z_\Phi(s, \chi)$ in the variable $t = q^{-s}$. The following formula is a variation of Proposition 1.4.4 given by Denef in [6], see also [28, Proposition 4.6]: for $u \in R^\times$ and $m \in \mathbb{Z}$,

$$(4.1) \quad \begin{aligned} E_\Phi(u\pi^{-m}) &= Z_\Phi(0, \chi_{\text{triv}}) + \text{Coeff}_{t^{m-1}} \frac{(t-q)Z_\Phi(s, \chi_{\text{triv}})}{(q-1)(1-t)} \\ &\quad + \sum_{\chi \neq \chi_{\text{triv}}} g_{\chi^{-1}} \chi(u) \text{Coeff}_{t^{m-c(\chi)}} Z_\Phi(s, \chi), \end{aligned}$$

where $c(\chi)$ is the conductor of χ , and g_χ denotes the Gaussian sum

$$g_\chi = (q-1)^{-1} q^{1-c(\chi)} \sum_{v \in (R/P^{c(\chi)})^\times} \chi(v) \Psi(v/\pi^{c(\chi)}).$$

By using the hypothesis $C_{f_l} \subset f_l^{-1}(0)$, $Z_\Phi(s, \chi)$ is a rational function identically zero for almost all χ (cf. Theorem 2), hence the series in the right side of (4.1) is a finite sum. The asymptotic expansion for $E_\Phi(z)$ is obtained by expanding the right side of (4.1) in partial fractions.

(2) The estimation for $|z|_K$ big enough is obtained as follows: there exist ρ, m_ρ (cf. Remark 2) such that for every pole λ in Theorem 3 (1),

$$|z|_K^\lambda (\log_q |z|_K)^\beta \leq C |z|_K^{-\rho} (\log_q |z|_K)^{m_\rho-1},$$

for $|z|_K$ big enough, and some constant C . The estimation of $|E_\Phi(z)|$, for $|z|_K > 1$, follows from the previous estimation by adjusting the constant C , since $|E_\Phi(z)|$ is upper bounded. \square

The estimation given in the second part of Theorem 3 is optimal, in the sense that there exists a Φ such that the constants ρ, m_ρ cannot be improved.

5. EXPONENTIAL SUMS ALONG p -ADIC SUBMANIFOLDS OF R_K^n

5.1. Some additional notation. From now on, we assume that all the $f_i(x)$ have coefficients in R_K , and put $U = R_K^n$, and set

$$V^{(j)}(R_K) := \{x \in R_K^n \mid f_i(x) = 0, i = 1, \dots, j\}$$

for $j = l-1, l$. Note that $V^{(l-1)}(R_K)$ is a closed submanifold of dimension $n-l+1 \geq 1$.

Let $\text{mod } P_K^m$ denote the canonical homomorphism $R_K^n \rightarrow (R_K/P_K^m)^n$, for $m \in \mathbb{N} \setminus \{0\}$. The image of $x \in R_K^n$ under this homomorphism is denoted by \bar{x} . We will call the image of $A \subseteq R_K^n$ by $\text{mod } P_K^m$, the *reduction mod P_K^m of A* , and it will be denoted as $A \text{ mod } P_K^m$. When we write $\bar{x} \in A \text{ mod } P_K^m$, we always assume without mentioning that $x \in A$. We will apply these definitions for A equal to $V^{(l-1)}(R_K)$.

For any polynomial g over R_K we denote by \bar{g} the polynomial over \bar{K} obtained by reducing each coefficient of g modulo P_K .

We define for $m \in \mathbb{N} \setminus \{0\}$ the set

$$V^{(l-1)}(R_K/P_K^m) = \{\bar{x} \in (R_K/P_K^m)^n \mid \text{ord}(f_i(x)) \geq m, i = 1, \dots, l-1\}.$$

We note that “ $\text{ord}(f_i(x)) \geq m$ ” is independent of the representative chosen to compute $\text{ord}(f_i(x))$. For $m = 1$, we write often $V^{(l-1)}(\bar{K})$ instead of $V^{(l-1)}(R_K/P_K)$. Note that

$$V^{(l-1)}(\bar{K}) = \left\{x \in \bar{K}^n \mid \bar{f}_i(x) = 0, i = 1, \dots, l-1\right\}.$$

We will say that $V^{(l-1)}(R_K)$ has *good reduction mod P_K* if

$$\text{rank}_{\bar{K}} \left[\frac{\partial \bar{f}_i}{\partial x_j}(x) \right]_{\substack{1 \leq i \leq l-1 \\ 1 \leq j \leq n}} = l-1, \text{ for every } x \in V^{(l-1)}(\bar{K}).$$

If $V^{(l-1)}(R_K)$ has good reduction mod P_K , the Hensel lemma implies that

$$V^{(l-1)}(R_K) \text{ mod } P_K^m = V^{(l-1)}(R_K/P_K^m)$$

for every $m \in \mathbb{N} \setminus \{0\}$.

5.2. Submanifolds with bad reduction mod P_K . In general the reduction $V^{(l-1)}(R_K) \bmod P_K$ has singular points. In order to deal with these points we use some ideas about Néron π -desingularization, see e.g. [2, Section 4], or [28, Proposition 2.4 and Lemma 2.5]. Since we did not find a suitable reference for our purposes, we prove below the required results.

Lemma 1. *Suppose that $V^{(l-1)}(R_K)$ is a closed submanifold of dimension $n - l + 1$. Let $x_0 \in V^{(l-1)}(R_K)$. Then, there exist a positive integer L and polynomials $f_{1,x_0}, \dots, f_{l-1,x_0} \in R_K[x_1, \dots, x_n]$ such that*

$$V^{(l-1)}(R_K) = \{x \in R_K^n \mid f_{i,x_0}(x) = 0, i = 1, \dots, l-1\},$$

and if we define

$$f_{i,x_0}(x_0 + \pi^L y) = \pi^{e_{i,x_0,L}} f_{i,x_0,L}(y),$$

with $f_{i,x_0,L}(y) \in R_K[y_1, \dots, y_n] \setminus P_K[y_1, \dots, y_n]$, for $i = 1, \dots, l-1$, then, the submanifold

$$V_{x_0,L}^{(l-1)}(R_K) := \{y \in R_K^n \mid f_{i,x_0,L}(y) = 0, i = 1, \dots, l-1\}$$

has good reduction mod P_K .

Proof. By applying a translation, we can assume that x_0 is the origin. We set

$$f_i(x) = a_{i,1}x_1 + \dots + a_{i,n}x_n + \text{higher degree terms},$$

for $i = 1, \dots, l-1$. The matrix $(a_{i,j})$ has rank $l-1$ over K because $V^{(l-1)}(R_K)$ is a closed submanifold of dimension $n - l + 1$. The announced polynomials f_{i,x_0} are linear combinations with coefficients in R_K of the f_i . These linear combinations are determined by the elementary row operations over R_K required to reduced the matrix $(a_{i,j})$ to its row echelon form. Since R_K is not a field some details are required. We select an entry a_{i_0,j_0} of $(a_{i,j})$ satisfying $\text{ord}(a_{i_0,j_0}) = \min_{i,j} \text{ord}(a_{i,j})$. Then by row and column interchanging one gets a new matrix having a_{i_0,j_0} in the position $(1,1)$. Thus we can assume that $i_0 = j_0 = 1$. In addition, by row interchanging we can assume that

$$\text{ord}(a_{2,1}) \leq \text{ord}(a_{3,1}) \leq \dots \leq \text{ord}(a_{l-1,1}).$$

We now can perform elementary row operations on R_K to obtain a matrix $(a'_{i,j})$ satisfying $a'_{1,1} = a_{1,1}$, $a'_{2,1} = \dots = a'_{l-1,1} = 0$. We now apply the previous procedure to $(a'_{i,j})$, $2 \leq i \leq l-1$, $1 \leq j \leq n$. By using this procedure, we construct a matrix $(b_{i,j})$ which is the row echelon form with $\text{rank}_K(b_{i,j}) = \text{rank}_K(a_{i,j}) = l-1$ and define

$$f_{i,x_0}(x) = b_{i,i}x_i + \dots + b_{i,n}x_n + \text{higher degree terms},$$

for $i = 1, \dots, l-1$. Note that

$$\text{ord}(b_{1,1}) \leq \text{ord}(b_{2,2}) \leq \dots \leq \text{ord}(b_{l-1,l-1})$$

with $b_{i,i} \neq 0$, $1 \leq i \leq l-1$, because $\text{rank}_K(b_{i,j}) = \text{rank}_K(a_{i,j}) = l-1$. In addition,

$$\text{ord}(b_{i,i}) \leq \text{ord}(b_{i,j}), \text{ for } i < j \leq n.$$

We set

$$L := \text{ord}(b_{l-1,l-1}) + 1 < \infty,$$

and

$$f_{i,x_0}(\pi^L y) := \pi^{e_{i,x_0,L}} f_{i,x_0,L}(y),$$

where $e_{i,x_0,L} = L + \text{ord}(b_{i,i})$, $f_{i,x_0,L}(y) = c_{i,i}x_i + c_{i,i+1}x_{i+1} + \dots + c_{i,n}x_n +$ (higher degree terms), $c_{i,i} = ac(b_{i,i}) \in R_K^\times$, for $i = 1, \dots, l-1$. Note that $\text{rank}_K(b_{i,j}) = \text{rank}_K(c_{i,j}) = l-1$, and that $\text{rank}_{\overline{K}}(\overline{c}_{i,j}) = l-1$.

Finally, since

$$\text{rank}_{\overline{K}}\left(\frac{\partial \overline{f_{i,x_0,L}}}{\partial y_j}(\overline{x})\right) = \text{rank}_{\overline{K}}(\overline{c}_{i,j}) = l-1,$$

for every $\overline{x} \in V_{x_0,L}^{(l-1)}(\overline{K})$, we conclude that $V_{x_0,L}^{(l-1)}(R_K)$ has good reduction mod P_K . \square

In the above proof, we can take for each $x' = x_0 + (P_K^{L+1})^n \cap V^{(l-1)}(R_K)$ the same L . Thus by the compactness of $V^{(l-1)}(R_K)$ there are only finitely many L involved, and consequently, by taking the maximum of these numbers, we can take L independently of x_0 . In this way we get the following result.

Proposition 1. *Assume that $V^{(l-1)}(R_K)$ is a closed submanifold of dimension $n-l+1$. Then, there exist a positive integer L such that for every $x_0 \in V^{(l-1)}(R_K)$, there exist equations $f_{1,x_0}(x) = \dots = f_{l-1,x_0}(x) = 0$ with coefficients in R_K defining $V^{(l-1)}(R_K)$ such that if we write $f_{i,x_0}(x_0 + \pi^L y) = \pi^{e_{i,x_0,L}} f_{i,x_0,L}(y)$, with $f_{i,x_0,L}(y) \in R_K[y_1, \dots, y_n] \setminus P_K[y_1, \dots, y_n]$, for $i = 1, \dots, l-1$, the submanifold $V_{x_0,L}^{(l-1)}(R_K) = \{y \in R_K^n \mid f_{i,x_0,L}(y) = 0, i = 1, \dots, l-1\}$ has good reduction mod P_K .*

5.3. Bounding $|E(z)|$. In this section, we use $\rho = \rho(V^{(l-1)}, f_l, \gamma_{GL})$ and $m_\rho = m_\rho(V^{(l-1)}, f_l, \gamma_{GL})$ which were defined before, see Remark 2.

We set for $z = u\pi^{-m} \in K$, with $u \in R^\times$ and $m \in \mathbb{N} \setminus \{0\}$, as in the introduction, the exponential sum

$$E(z) = q^{-m(n-l+1)} \sum_{\overline{x} \in V^{(l-1)}(R_K) \bmod P_K^m} \Psi(z f_l(x)).$$

Theorem 4. *Assume that $C_{f_l} \subset f_l^{-1}(0)$. Then there exists a constant C such that*

$$|E(z)| \leq C |z|_K^{-\rho} (\log_q |z|_K)^{m_\rho-1},$$

for $|z|_K > 1$.

Proof. Consider first the case in which $V^{(l-1)}(R_K)$ has good reduction mod P_K . By applying Lemma 5.5 in [29],

$$E(z) = \int_{V^{(l-1)}(R_K)} \Psi(z f_l(x)) \mid \gamma_{GL} \mid,$$

now, the announced estimation follows from Theorem 3.

We now consider the general case. We use Proposition 1 to reduce the estimation of $|E(z)|$ to the estimation of several exponential sums along several submanifolds with good reduction mod P_K , as follows. There exists a positive integer L such that for every $x \in V^{(l-1)}(R_K)$, there exist equations $f_{1,x}, \dots, f_{l-1,x}$ of $V^{(l-1)}(R_K)$ such that if we write $f_{i,x}(x + \pi^L y) = \pi^{e_{i,x,L}} f_{i,x,L}(y)$ for $i = 1, \dots, l-1$, with $f_{i,x,L}(y) \in R_K[y_1, \dots, y_n] \setminus P_K[y_1, \dots, y_n]$, the manifold

$$V_{x,L}^{(l-1)}(R_K) = \{y \in R_K^n \mid f_{i,x,L}(y) = 0, i = 1, \dots, l-1\}$$

has good reduction mod P_K . Take such an integer L . For $x \in V^{(l-1)}(R_K)$, we write $f_l(x + \pi^L y) = f_l(x) + \pi^{e_{l,x,L}} f_{l,x,L}(y)$, with $f_{l,x,L}(y) \in R_K[y_1, \dots, y_n] \setminus P_K[y_1, \dots, y_n]$. Note that $e_{l,x,L} \geq L$.

For $z = u\pi^{-m}$, with $u \in R_K^\times$ and $m > L$, we get

$$\begin{aligned} & q^{m(n-l+1)} E(u\pi^{-m}) \\ &= \sum_{\bar{x} \in V^{(l-1)}(R_K) \bmod P_K^m} \Psi(u f_l(x) / \pi^m) \\ &= \sum_{\bar{x} \in V^{(l-1)}(R_K) \bmod P_K^L} \sum_{\bar{y} \in V_{x,L}^{(l-1)}(R_K) \bmod P_K^{m-L}} \Psi(u f_l(x + \pi^L y) / \pi^m) \\ &= \sum_{\bar{x} \in V^{(l-1)}(R_K) \bmod P_K^L} \Psi\left(\frac{u f_l(x)}{\pi^m}\right) \sum_{\bar{y} \in V_{x,L}^{(l-1)}(R_K) \bmod P_K^{m-L}} \Psi\left(\frac{u \pi^{e_{l,x,L}-L} f_{l,x,L}(y)}{\pi^{m-L}}\right). \end{aligned}$$

Therefore,

$$|E(u\pi^{-m})| \leq C_0 \max_{\bar{x} \in V^{(l-1)}(R_K) \bmod P_K^L} \left| \sum_{\bar{y} \in V_{x,L}^{(l-1)}(R_K) \bmod P_K^{m-L}} \Psi\left(\frac{u \pi^{e_{l,x,L}-L} f_{l,x,L}(y)}{\pi^{m-L}}\right) \right|,$$

where $C_0 = \#(V^{(l-1)}(R_K) \bmod P_K^L)$.

Since $\theta_x : K^n \rightarrow K^n$, with $\theta_x(y) = x + \pi^L y$ is a K -analytic isomorphism for any $x \in R_K^n$, and $\theta_x(V_{x,L}^{(l-1)}(R_K)) = V^{(l-1)}(R_K) \cap (x + (P_K^L)^n)$, we have

$$\begin{aligned} (5.1) \quad \rho &= \rho(V^{(l-1)}, f_l, \gamma_{GL}) \\ &\leq \rho(V_{x,L}^{(l-1)}, f_{l,x,L}, \gamma_{GL}), \text{ for any } \bar{x} \in V^{(l-1)}(R_K) \bmod P_K^L, \end{aligned}$$

$$(5.2) \quad \rho = \rho(V_{x,L}^{(l-1)}, f_{l,x,L}, \gamma_{GL}), \text{ for some } \bar{x} \in V^{(l-1)}(R_K) \bmod P_K^L,$$

$$\begin{aligned} (5.3) \quad m_\rho &= m_\rho(V^{(l-1)}, f_l, \gamma_{GL}) \\ &\geq m_\rho(V_{x,L}^{(l-1)}, f_{l,x,L}, \gamma_{GL}), \text{ for any } \bar{x} \in V^{(l-1)}(R_K) \bmod P_K^L. \end{aligned}$$

We now note that all the $V_{x,L}^{(l-1)}(R_K)$ have good reduction mod P_K , and then by applying the estimation given at the beginning of the proof, we have

$$(5.4) \quad |E(u\pi^{-m})| \leq C_0 q^{-m\rho} m^{m_\rho-1}, \text{ for } m \geq L \geq 1.$$

Finally, since

$$\begin{aligned} |E(u\pi^{-m})| &\leq q^{-m(n-l+1)} \#(V^{(l-1)}(R_K) \bmod P_K^m) \\ &\leq q^{m(l-1)}, \end{aligned}$$

for every m , we can replace C_0 by

$$C := \max \left\{ \{C_0\} \cup \left\{ \frac{q^{m(l-1)}}{q^{-m\rho} m^{m_\rho-1}} \mid 1 \leq m \leq L-1 \right\} \right\},$$

in (5.4), and thus the estimation holds for all $m \geq 1$. \square

6. POINCARÉ SERIES AND POLYNOMIAL CONGRUENCES ALONG p -ADIC
SUBMANIFOLDS OF R_K^n

We define for $m \in \mathbb{N}$ the number $N_m = N_m(V^{(l-1)}, f_l)$ as

$$(6.1) \quad \begin{cases} \#(\{\bar{x} \in V^{(l-1)}(R_K) \bmod P_K^m \mid \text{ord}(f_l(x)) \geq m\}) & \text{if } m \geq 1 \\ 1 & \text{if } m = 0. \end{cases}$$

Note that $\text{ord}(f_l(x)) \geq m$, if and only if $f_l(x) \equiv 0 \bmod P_K^m$, and therefore, the N_m give the number of solutions of a polynomial congruence along the submanifold $V^{(l-1)}(R_K)$. We also define

$$P(t) := P(t, V^{(l-1)}, f_l) = \sum_{m=0}^{\infty} q^{-m(n-l+1)} N_m t^m.$$

If $V^{(l-1)}(R_K)$ has good reduction mod P_K , then

$$N_m = \#(\{\bar{x} \in (R_K/P_K^m)^n \mid f_1(x) \equiv f_2(x) \equiv \dots \equiv f_l(x) \equiv 0 \bmod P_K^m\}).$$

In the following theorem, we prove the rationality of $P(t)$ and give an upper bound for the N_m .

Theorem 5. (1) $P(t)$ is a rational function of q^{-s} . (2) There exists a constant C such that

$$N_m \leq C q^{(n-l+1-\rho)m} m^{m_\rho-1},$$

for all $m \geq 1$.

Proof. We first prove (1) and (2) assuming that $V^{(l-1)}(R_K)$ has good reduction mod P_K . By Lemma 5.3 in [29],

$$(6.2) \quad P(t) = \frac{1 - t Z_\Phi(s, \chi_{\text{triv}}, V^{(l-1)}, f_l)}{1 - t},$$

with $t = q^{-s}$ and Φ the characteristic function of R_K^n . The rationality follows from Theorem 2 and the upper bound follows from (6.2) by expanding the right in partial fractions.

For the general case, we use Proposition 1 as in the proof of Theorem 4. For $m > L$, one gets

$$N_m = \sum_{\bar{x} \in V^{(l-1)}(R_K) \bmod P_K^L} \# \{\bar{y} \in V_{x,L}^{(l-1)}(R_K) \bmod P_K^{m-L} \mid f_l(x + \pi^L y) \equiv 0 \bmod P_K^m\}.$$

If $f_l(x + \pi^L y) = 0$ has no solution in R_K^n , then for m big enough, the congruence $f_l(x + \pi^L y) \equiv 0 \bmod P_K^m$ has no solutions. Thus, there exists a natural number $m_0 \geq L$ such that if $m \geq m_0$, then N_m equals

$$\sum_{\bar{x} \in V^{(l)}(R_K) \bmod P_K^L} \# \{\bar{y} \in V_{x,L}^{(l-1)}(R_K) \bmod P_K^{m-L} \mid \begin{matrix} f_l(x + \pi^L y) \equiv \\ 0 \bmod P_K^m \end{matrix} \}$$

$$\begin{aligned}
&= \sum_{\bar{x} \in V^{(l)}(R_K) \bmod P_K^L} \# \{ \bar{y} \in V_{x,L}^{(l-1)}(R_K) \bmod P_K^{m-L} \mid \begin{array}{l} \pi^{e_{l,x,L}} f_{l,x,L}(y) \equiv \\ 0 \bmod P_K^m \end{array} \} \\
(6.3) \quad &= \sum_{\bar{x} \in V^{(l)}(R_K) \bmod P_K^L} \# \{ \bar{y} \in V_{x,L}^{(l-1)}(R_K) \bmod P_K^{m-L} \mid \begin{array}{l} \pi^{e_{l,x,L}-L} f_{l,x,L}(y) \equiv \\ 0 \bmod P_K^{m-L} \end{array} \}.
\end{aligned}$$

We now prove (1) as follows: since all the submanifolds $V_{x,L}^{(l-1)}(R_K)$ have good reduction mod P_K , by the rationality of $P\left(t, V_{x,L}^{(l-1)}, \pi^{e_{l,x,L}-L} f_{l,x,L}\right)$, there exists a constant $M_0(x)$ such that all the numbers $N_m\left(V_{x,L}^{(l-1)}, \pi^{e_{l,x,L}-L} f_{l,x,L}\right)$ satisfy a linear recurrence for $m > M_0(x)$ and for all $\bar{x} \in V^{(l)}(R_K) \bmod P_K^L$. Therefore, by (6.3), the numbers N_m satisfy a linear recurrence for m big enough, and the corresponding Poincaré series is rational.

Finally, we establish the announced bound for the N_m . Since the bound holds for the $N_m\left(V_{x,L}^{(l-1)}, \pi^{e_{l,x,L}-L} f_{l,x,L}\right)$, using (6.3) and (5.1)-(5.3), one gets

$$(6.4) \quad N_m \leq C_0 q^{(n-l+1-\rho)m} m^{m_\rho-1},$$

for all $m > M_1 > 1$, for some positive constants C_0 and $M_1 \in \mathbb{N}$. We now take

$$C := \max \left\{ \{C_0\} \cup \left\{ \frac{N_m}{q^{(n-l+1-\rho)m} m^{m_\rho-1}} \mid 1 \leq m \leq M_1 \right\} \right\}.$$

Finally, we can replace C_0 by C in (6.4), to obtain a bound valid for any $m \geq 1$. \square

Remark 3. Let $h(x_1, \dots, x_n)$, $g_i(x_1, \dots, x_n)$, $i = 1, \dots, l$ be non-constant polynomials with coefficients in R_K . Set

$$S(R_K) = \{x \in R_K^n \mid g_i(x) = 0, i = 1, \dots, l\}.$$

Assume that $S(R_K)$ is a K -analytic subset of R_K^n of dimension d , see [21] for this notion, and that $h|_{S(R_K)} \neq 0$. We define $N_m(S, h)$ as in (6.1), and

$$P(t, S, h) = \sum_{m=0}^{\infty} q^{-md} N_m(S, h) t^m.$$

A general result due to Denef implies the rationality of $P(t, S, h)$, see [8, Theorem 1.6.1]. Indeed, to see this, consider the set of all y in R_K^n , such that there exists x in R_K^n such that

$$g_1(x) = 0, \dots, g_l(x) = 0, \text{ and } \text{ord}(x - y) \geq m, \text{ and } \text{ord}(h(y)) \geq m.$$

This set depends on a positive integer m , and is definable by a formula in predicate logic. By considering its measure and applying Theorem 1.6.1 of Denef in [8], the rationality of $P(t, S, h)$ is established. Note that this result does not give information about the ‘poles’ of $P(t, S, h)$, and thus estimations for $N_m(S, h)$ cannot be obtained from it directly.

7. $Z_\Phi(\omega, V^{(l-1)}, f_l)$ AS A LIMIT OF INTEGRALS OVER K^n

As before, we take converging power series $f_i \in K[[x_1, \dots, x_n]]$ for $i = 1, \dots, l-1$ on an open and compact subset U of K^n , and assume that $V^{(l-1)} = \{x \in U \mid f_i(x) = 0, \text{ for } 1 \leq i \leq l-1\}$ is a closed K -analytic submanifold of U . In this section, we specialize Θ to a Gel'fand-Leray form γ_{GL} on $V^{(l-1)}$. We consider on K^n the measure $|dx|$ associated to the differential form $dx = dx_1 \wedge \dots \wedge dx_n$, which is the Haar measure on K^n so normalized that R_K^n has measure 1.

The second author proved in [29, Lemma 2.5] that

$$Z_\Phi(\omega, V^{(l-1)}, f_l) = \int_{K^n} \Phi(x) \delta(f_1(x), \dots, f_{l-1}(x)) \omega(f_l(x)) |dx|,$$

for $\omega \in \Omega_0(K^\times)$, where δ is the Dirac delta function. We recall that for a Bruhat-Schwartz function θ ,

$$\int_{K^n} \theta(x) \delta(f_1(x), \dots, f_{l-1}(x)) |dx| = \lim_{r \rightarrow +\infty} \int_{K^n} \theta(x) \delta_r(f_1(x), \dots, f_{l-1}(x)) |dx|,$$

where the functions δ_r for $r \in \mathbb{N}$ are defined by

$$\delta_r(u) = \begin{cases} 0 & \text{if } u \notin (\pi^r R_K)^{l-1} \\ q^{r(l-1)} & \text{if } u \in (\pi^r R_K)^{l-1}. \end{cases}$$

Note that $\omega(f_l(x))$ is not a Bruhat-Schwartz function.

From an intuitive point of view, the zeta functions considered here are defined by concentrating classical Igusa's zeta functions on submanifolds, see [11], [29]. In this way, several possible definitions for the local zeta function along a submanifold appear, and then, a natural question is to know if these definitions are equivalent. The next proposition answers a question posed in [29].

Theorem 6. *For $\omega \in \Omega_0(K^\times)$,*

$$Z_\Phi(\omega, V^{(l-1)}, f_l) = \lim_{r \rightarrow +\infty} \int_{K^n} \Phi(x) \delta_r(f_1(x), \dots, f_{l-1}(x)) \omega(f_l(x)) |dx|.$$

Proof. Let $b \in V^{(l-1)}(K)$. After a possibly reordering of the coordinates x_1, \dots, x_n , there exists a local coordinate change around b of the form $y = (y_1, \dots, y_n) = \phi(x)$, with

$$y_i := \begin{cases} f_i(x) & \text{if } i = 1, \dots, l-1 \\ x_i - b_i & \text{if } i = l, \dots, n, \end{cases}$$

since $V^{(l-1)}(K)$ is a submanifold. For d large enough, we have an open and compact neighborhood W of b such that $\phi : W \rightarrow (\pi^d R_K)^n$ is a K -analytic isomorphism satisfying $|J(x)|_K = |J(b)|_K$, for any $x \in W$, where $J(x)$ is the Jacobian of ϕ . It is sufficient to prove the theorem for Φ equal to the characteristic function of a such set W , since there exists finitely many disjoint subsets W as above on which Φ is constant, and which cover $V^{(l-1)}$. From now on, we suppose that Φ is the

characteristic function of W . By using $y = \phi(x)$ as a change of variables,

$$\begin{aligned} I_r(\omega) &:= \int_{K^n} \Phi(x) \delta_r(f_1(x), \dots, f_{l-1}(x)) \omega(f_l(x)) |dx| \\ &= |J(b)|_K^{-1} \int_{(\pi^d R_K)^n} \delta_r(y_1, \dots, y_{l-1}) \omega(\tilde{f}_l(y_1, \dots, y_n)) |dy_1 \dots dy_n|, \end{aligned}$$

where $\tilde{f}_l := f_l \circ \phi^{-1}$. Since $\delta_r(y_1, \dots, y_{l-1}) = q^{r(l-1)}$ if and only if $y_i \in \pi^r R_K$ for $i = 1, \dots, l-1$, and by assuming that $r \geq d$, we obtain

$$\begin{aligned} I_r(\omega) &= |J(b)|_K^{-1} q^{r(l-1)} \int_{(\pi^r R_K)^{l-1} \times (\pi^d R_K)^{n-l+1}} \omega(\tilde{f}_l(y_1, \dots, y_n)) |dy_1 \dots dy_n| \\ &= |J(b)|_K^{-1} \int_{R_K^{l-1} \times (\pi^d R_K)^{n-l+1}} \omega(\tilde{f}_l(\pi^r y_1, \dots, \pi^r y_{l-1}, y_l, \dots, y_n)) |dy_1 \dots dy_n|. \end{aligned}$$

Finally, by using that $\omega \in \Omega_0(K^\times)$ and the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{r \rightarrow +\infty} I_r(\omega) &= |J(b)|_K^{-1} \int_{(\pi^d R_K)^{n-l+1}} \omega(\tilde{f}_l(0, \dots, 0, y_l, \dots, y_n)) |dy_l \dots dy_n| \\ &= \int_{V^{(l-1)}} \Phi(x) \omega(f_l(x)) |\gamma_{GL}|. \end{aligned}$$

□

Remark 4. *The previous result is still valid if we replace $\omega(f_l(x))$ by any continuous complex-valued function, in particular,*

$$\begin{aligned} E_\Phi(z) &= \int_{V^{(l-1)}} \Phi(x) \Psi(z f_l(x)) |\gamma_{GL}| \\ &= \int_{K^n} \Phi(x) \delta(f_1(x), \dots, f_{l-1}(x)) \Psi(z f_l(x)) |dx| \\ &= \lim_{r \rightarrow +\infty} \int_{K^n} \Phi(x) \delta_r(f_1(x), \dots, f_{l-1}(x)) \Psi(z f_l(x)) |dx|. \end{aligned}$$

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